Rarefaction waves

$$u_t + f(u)_x = 0 \qquad \begin{array}{l} u \in \mathbb{R}^n \\ f \in \mathbb{R}^n \end{array}$$

Assume system is strictly hyperbolic, i.e. $df(u)$ has n distinct real e.v.

Note that the equation is scale invariant, i.e. if u is a solution, then also

$$v = v(x,t) = u(hx, ht) \quad h \text{ const.}$$

will be a solution since

$$v_t + f(v)_x = u_t h + df'(u) u_x h$$
$$= h(u_t + f(u)_x) = 0$$

Then we can assume that

$$u(x,t) = w(x/t) \qquad \xi = x/t$$

Insert this in the equation.

Then:

$$0 = u_t + f(u)_x = u_t + df(u)\,u_x$$

$$= \dot{w}\left(-x/t^2\right) + df(w)\,\dot{w}\,\frac{1}{t}$$

$$= \frac{x}{t}\left(df(w)\dot{w} - \frac{x}{t}\,\dot{w}\right)$$

or

$$\underline{df(w)\,\dot{w} = \xi\,\dot{w}}$$

Scalar case: If $\dot{w} \neq 0$, we get $df(w) = \xi$ and we got $w = (df)^{-1}(\xi)$ which was the formula for rare-faction <u>waves</u>. $\circ$

In case of systems, we have

$$df(w)\,\dot{w} = \xi\,\dot{w}$$

$$df(w)\,r_j = \lambda_j\,r_j$$

Thus: $\begin{cases} \underline{\dot{w} = r_j(w)} \\ \underline{\lambda_j(w) = \xi} \end{cases}$ ← system of ODEs

Consider the Riemann problem

$$u(x,0) = \begin{cases} u_l & x<0 \\ u_r & x>0 \end{cases}$$

Then we have the following

$\underline{w(\lambda_j(u_l)) = u_l}$

$\underline{w(\lambda_j(u_r)) = u_r}$

We have:

$$\lambda_j(w(\xi)) = \xi$$

The derivative w.r.t. ξ:

$$\underline{1} = \frac{d}{d\xi}\xi = \frac{d}{d\xi}\lambda_j(w(\xi))$$

$$= \nabla\lambda_j(w(\xi))\cdot\dot{w}(\xi)$$

$$= \underline{\nabla\lambda_j\cdot r_j(\xi)}$$

If $\nabla\lambda_j\cdot r_j \neq 0$, say $\nabla\lambda_j\cdot r_j = c(\xi)$

we can define $\tilde{r}_j = \frac{1}{c} r_j$. Then

$$\nabla\lambda_j\cdot\tilde{r}_j = 1$$

$\nabla\lambda_j\cdot r_j = 1$ genuinely nonlinear GN

$\nabla\lambda_j\cdot r_j = 0$ linearly degenerate LD

We will assume that the system for each j is either GN or LD for all points in the domain.

For the Euler equations one of the families is LD!

We can summarize what we have found in the following thm:

<u>Thm 5.6</u> Assume system is strictly hyperbolic in the domain $u \in D \subset \mathbb{R}^n$. Consider family j and assume this family is GN, and assume that $\nabla \lambda_j(u) \cdot r_j(u) = 1$ in D. Let $u_\ell \in D$. Then there exists a curve $R_j(u_\ell)$ which emanates from u_ℓ s.t. for any $u_r \in R_j(u_\ell)$ the solution to the RP

$$u\big|_{t=0} = \begin{cases} u_\ell & x < 0 \\ u_r & x > 0 \end{cases}$$

is given by

$$u(x,t) = \begin{cases} u_\ell, & x < \lambda_j(u_\ell)t \\ w(x/t), & \lambda_j(u_\ell)t < x < \lambda_j(u_r)t \\ u_r, & x > \lambda_j(u_r)t \end{cases}$$

where w satisfies:

$$\dot{w} = r_j(w), \quad \lambda_j(w(\xi)) = \xi$$
$$w(\lambda_j(u_\ell)) = u_\ell$$
$$w(\lambda_j(u_r)) = u_r$$

Pf. Solve eq'n $\begin{cases} \dot{w} = r_j(w) \\ w(\lambda_j(u_\ell)) = u_\ell \end{cases}$

locally. We have $1 = \nabla\lambda_j \cdot r_j$ which gives $\xi = \lambda_j(w(\xi))$

Denote the orbit of this curve as R_j. Along this curve we have $w(\lambda_j(u_r)) = u_r$.